

On the properties of iterated binomial transforms for the Padovan and Perrin matrix sequences

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Abstract

In this study, we apply " r " times the binomial transform to the Padovan and Perrin matrix sequences. Also, the Binet formulas, summations, generating functions of these transforms are found using recurrence relations. Finally, we give the relationships of between iterated binomial transforms for Padovan and Perrin matrix sequences.

Keywords: Padovan matrix sequence, Perrin matrix sequence, iterated binomial transform.

Ams Classification: 11B65, 11B83.

1 Introduction and Preliminaries

There are so many studies in the literature that concern about the special number sequences such as Fibonacci, Lucas, Pell, Padovan and Perrin (see, for example [1, 2, 3, 4], and the references cited therein). In Fibonacci numbers, there clearly exists the term Golden ratio which is defined as the ratio of two consecutive of Fibonacci numbers that converges to $\alpha = \frac{1+\sqrt{5}}{2}$. It is also clear that the ratio has so many applications in, specially, Physics, Engineering, Architecture, etc.[5, 6]. In a similar manner, the ratio of two consecutive Padovan and Perrin numbers converges to

$$\alpha_P = \sqrt[3]{\frac{1}{2} + \frac{1}{6}\sqrt{\frac{23}{3}}} + \sqrt[3]{\frac{1}{2} - \frac{1}{6}\sqrt{\frac{23}{3}}}$$

that is named as *Plastic constant* and was firstly defined in 1924 by Gérard Cordonnier. He described applications to architecture and illustrated the use of the Plastic constant in many buildings.

Although the study of Perrin numbers started in the beginning of 19. century under different names, the master study was published in 2006 by Shannon et al.

in [4]. The authors defined the Perrin $\{R_n\}_{n \in \mathbb{N}}$ and Padovan $\{P_n\}_{n \in \mathbb{N}}$ sequences as in the forms

$$R_{n+3} = R_{n+1} + R_n, \text{ where } R_0 = 3, R_1 = 0, R_2 = 2 \quad (1.1)$$

and

$$P_{n+3} = P_{n+1} + P_n, \text{ where } P_0 = P_1 = P_2 = 1, \quad (1.2)$$

respectively.

On the other hand, the matrix sequences have taken so much interest for different type of numbers [7, 8, 9, 10]. For instance, in [7], authors defined new matrix generalizations for Fibonacci and Lucas numbers, and using essentially a matrix approach they showed some properties of these matrix sequences. In [8], Gulec and Taskara gave new generalizations for (s, t) -Pell and (s, t) -Pell Lucas sequences for Pell and Pell–Lucas numbers. Considering these sequences, they defined the matrix sequences which have elements of (s, t) -Pell and (s, t) -Pell Lucas sequences. Also, they investigated their properties. In [9], authors defined a new sequence in which it generalizes (s, t) -Fibonacci and (s, t) -Lucas sequences at the same time. After that, by using it, they established generalized (s, t) -matrix sequence. Finally, they presented some important relationships among this new generalization, (s, t) -Fibonacci and (s, t) -Lucas sequences and their matrix sequences. Moreover, in [10], authors develop the matrix sequences that represent Padovan and Perrin numbers and examined their properties.

In [10], for $n \geq 0$, authors defined Padovan and Perrin matrix sequences as in the forms

$$\mathcal{P}_{n+3} = \mathcal{P}_{n+1} + \mathcal{P}_n, \quad (1.3)$$

where

$$\mathcal{P}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathcal{P}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \mathcal{P}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

and

$$\mathcal{R}_{n+3} = \mathcal{R}_{n+1} + \mathcal{R}_n, \quad (1.4)$$

where

$$\mathcal{R}_0 = \begin{pmatrix} 4 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix}, \mathcal{R}_1 = \begin{pmatrix} -3 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{pmatrix}, \mathcal{R}_2 = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & -1 \\ -1 & 0 & 3 \end{pmatrix}.$$

Proposition 1.1 [10] *Let us consider $n, m \geq 0$, the following properties are hold:*

- $\mathcal{P}_m \mathcal{P}_n = \mathcal{P}_{n+m},$
- $\mathcal{P}_m \mathcal{R}_n = \mathcal{R}_n \mathcal{P}_m = \mathcal{R}_{n+m}.$

In addition, some matrix based transforms can be introduced for a given sequence. Binomial transform is one of these transforms and there is also other ones such as rising and falling binomial transforms(see [11, 12, 16]). Given an integer sequence $X = \{x_0, x_1, x_2, \dots\}$, the binomial transform B of the sequence X , $B(X) = \{b_n\}$, is given by

$$b_n = \sum_{i=0}^n \binom{n}{i} x_i.$$

In [13, 15], authors gave the application of the several class of transforms to the k -Fibonacci and k -Lucas sequence. In [14], the authors applied the binomial transforms to the Padovan (\mathcal{P}_n) and Perrin matrix sequences (\mathcal{R}_n) .

Proposition 1.2 [14] For $n > 0$,

i) Recurrence relation of sequences $\{b_n\}$ is

$$b_{n+2} = 3b_{n+1} - 2b_n + b_{n-1}, \quad (1.5)$$

with initial conditions $b_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $b_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $b_2 = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & 2 \end{pmatrix}$.

ii) Recurrence relation of sequences $\{c_n\}$ is

$$c_{n+2} = 3c_{n+1} - 2c_n + c_{n-1}, \quad (1.6)$$

with initial conditions $c_0 = \begin{pmatrix} 4 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix}$, $c_1 = \begin{pmatrix} 1 & 3 & -1 \\ -1 & 0 & 3 \\ 3 & 2 & 0 \end{pmatrix}$ and $c_2 = \begin{pmatrix} 0 & 3 & 2 \\ 2 & 2 & 3 \\ 3 & 5 & 2 \end{pmatrix}$.

iii) For $n, m \geq 0$, we have

$$b_n b_m = b_{n+m},$$

where $n \leq m$,

iv) For $n, m \geq 0$, we have

$$b_n c_m = c_n b_m = c_{n+m}.$$

Falcon [17] studied the iterated application of the some Binomial transforms to the k -Fibonacci sequence. For example, author obtained recurrence relation of the iterated binomial transform for k -Fibonacci sequence

$$a_{k,n+1}^{(r)} = (2r+k) a_{k,n}^{(r)} - (r^2 + kr - 1) a_{k,n-1}^{(r)}, \quad a_{k,0}^{(r)} = 0 \text{ and } a_{k,1}^{(r)} = 1.$$

Yilmaz and Taskara [18] studied the iterated application of the some Binomial transforms to the k -Lucas sequence. For example, author obtained recurrence relation of the iterated binomial transform for k -Lucas sequence

$$b_{k,n+1}^{(r)} = (2r+k)b_{k,n}^{(r)} - (r^2+kr-1)b_{k,n-1}^{(r)}, \quad b_{k,0}^{(r)} = 2 \text{ and } b_{k,1}^{(r)} = 2r+k.$$

Motivated by [10,14,17,18], the goal of this paper is to apply iteratly the binomial transforms to the Padovan (\mathcal{P}_n) and Perrin matrix sequences (\mathcal{R}_n) . Also, the binet formulas, summations, generating functions of these transforms are found by recurrence relations. Finally, it is illustrated the relations between of this transforms by deriving new formulas.

2 Iterated binomial transforms of the Padovan and Perrin matrix sequences

In this section, we will mainly focus on iterated binomial transforms of the Padovan and Perrin matrix sequences to get some important results. In fact, we will also present the recurrence relations, binet formulas, summations, generating functions of these transforms.

The iterated binomial transforms of the Padovan (\mathcal{P}_n) and Perrin matrix sequences (\mathcal{R}_n) are demonstrated by $B^{(r)} = \{b_n^{(r)}\}$ and $C^{(r)} = \{c_n^{(r)}\}$, where $b_n^{(r)}$ and $c_n^{(r)}$ are obtained by applying "r" times the binomial transform to the Padovan and Perrin matrix sequences. It are obvious that $b_0^{(r)} = \mathcal{P}_0$, $b_1^{(r)} = r\mathcal{P}_0 + \mathcal{P}_1$, $b_2^{(r)} = r^2\mathcal{P}_0 + 2r\mathcal{P}_1 + \mathcal{P}_2$ and $c_0^{(r)} = \mathcal{R}_0$, $c_1^{(r)} = r\mathcal{R}_0 + \mathcal{R}_1$, $c_2^{(r)} = r^2\mathcal{R}_0 + 2r\mathcal{R}_1 + \mathcal{R}_2$.

The following lemma will be key of the proof of the next theorems.

Lemma 2.1 *For $n \geq 0$ and $r \geq 1$, the following equalities are hold:*

$$i) \quad b_{n+2}^{(r)} = \sum_{j=0}^{n+1} \binom{n+1}{j} \left(b_j^{(r-1)} + b_{j+1}^{(r-1)} \right),$$

$$ii) \quad c_{n+2}^{(r)} = \sum_{j=0}^{n+1} \binom{n+1}{j} \left(c_j^{(r-1)} + c_{j+1}^{(r-1)} \right).$$

Proof. Firstly, in here we will just prove *i)*, since *ii)* can be thought in the same manner with *i)*.

i) By using definition of binomial transform, we obtain

$$\begin{aligned} b_{n+2}^{(r)} &= \sum_{j=0}^{n+2} \binom{n+2}{j} b_j^{(r-1)} \\ &= \sum_{j=1}^{n+2} \binom{n+2}{j} b_j^{(r-1)} + b_0^{(r-1)}. \end{aligned}$$

And by taking account the well known binomial equality

$$\binom{n+2}{i} = \binom{n+1}{i} + \binom{n+1}{i-1},$$

we have

$$\begin{aligned} b_{n+2}^{(r)} &= \sum_{j=1}^{n+1} \binom{n+1}{j} b_j^{(r-1)} + \sum_{j=1}^{n+2} \binom{n+1}{j-1} b_j^{(r-1)} + b_0^{(r-1)} \\ &= \sum_{j=0}^{n+1} \binom{n+1}{j} b_j^{(r-1)} + \sum_{j=0}^{n+1} \binom{n+1}{j} b_{j+1}^{(r-1)} \\ &= \sum_{j=0}^{n+1} \binom{n+1}{j} (b_j^{(r-1)} + b_{j+1}^{(r-1)}), \end{aligned}$$

which is desired result.

■

From above Lemma, note that:

- b_{n+2} is also can be written as $b_{n+2}^{(r)} = b_{n+1}^{(r)} + \sum_{i=0}^n \binom{n}{i} (b_{j+1}^{(r-1)} + b_{j+2}^{(r-1)})$,
- c_{n+2} is also can be written as $c_{n+2}^{(r)} = c_{n+1}^{(r)} + \sum_{i=0}^n \binom{n}{i} (c_{j+1}^{(r-1)} + c_{j+2}^{(r-1)})$.

Theorem 2.1 For $n \geq 0$ and $r \geq 1$, the recurrence relations of sequences $\{b_n^{(r)}\}$ and $\{c_n^{(r)}\}$ are

$$b_{n+2}^{(r)} = 3rb_{n+1}^{(r)} - (3r^2 - 1)b_n^{(r)} + (r^3 - r + 1)b_{n-1}^{(r)}, \quad (2.1)$$

$$c_{n+2}^{(r)} = 3rc_{n+1}^{(r)} - (3r^2 - 1)c_n^{(r)} + (r^3 - r + 1)c_{n-1}^{(r)}, \quad (2.2)$$

with initial conditions $b_0^{(r)} = \mathcal{P}_0$, $b_1^{(r)} = r\mathcal{P}_0 + \mathcal{P}_1$, $b_2^{(r)} = r^2\mathcal{P}_0 + 2r\mathcal{P}_1 + \mathcal{P}_2$ and $c_0^{(r)} = \mathcal{R}_0$, $c_1^{(r)} = r\mathcal{R}_0 + \mathcal{R}_1$, $c_2^{(r)} = r^2\mathcal{R}_0 + 2r\mathcal{R}_1 + \mathcal{R}_2$.

Proof. Similarly the proof of the Lemma 2.1, only the first case, the equation (2.1) will be proved. We will omit the equation (2.2) since the proofs will not be different.

The proof will be done by induction steps on r and n .

First of all, for $r = 1$, from the i) condition of Proposition 1.2, it is true $b_{n+2} = 3b_{n+1} - 2b_n + b_{n-1}$.

Let us consider definition of iterated binomial transform, then we have

$$b_3^{(r)} = (r^3 + 1)\mathcal{P}_0 + (3r^2 + 1)\mathcal{P}_1 + 3r\mathcal{P}_2.$$

The initial conditions are

$$b_0^{(r)} = \mathcal{P}_0, b_1^{(r)} = r\mathcal{P}_0 + \mathcal{P}_1, b_2^{(r)} = r^2\mathcal{P}_0 + 2r\mathcal{P}_1 + \mathcal{P}_2.$$

Hence, for $n = 1$, the equality (2.1) is true, that is $b_3^{(r)} = 3rb_2^{(r)} - (3r^2 - 1)b_1^{(r)} + (r^3 - r + 1)b_0^{(r)}$.

Actually, by assuming the equation in (2.1) holds for all $(r-1, n)$ and $(r, n-1)$, that is,

$$b_{n+2}^{(r-1)} = 3(r-1)b_{n+1}^{(r-1)} - (3(r-1)^2 - 1)b_n^{(r-1)} + ((r-1)^3 - (r-1) + 1)b_{n-1}^{(r-1)},$$

and

$$b_{n+1}^{(r)} = 3rb_n^{(r)} - (3r^2 - 1)b_{n-1}^{(r)} + (r^3 - r + 1)b_{n-2}^{(r)}.$$

Then, we need to show that it is true for (r, n) . That is,

$$b_{n+2}^{(r)} = 3rb_{n+1}^{(r)} - (3r^2 - 1)b_n^{(r)} + (r^3 - r + 1)b_{n-1}^{(r)}.$$

Let us label $b_{n+2}^{(r)} = 3rb_{n+1}^{(r)} - (3r^2 - 1)b_n^{(r)} + (r^3 - r + 1)b_{n-1}^{(r)}$ by *RHS*. Hence, we can write

$$\begin{aligned} RHS &= 3r \sum_{j=0}^{n+1} \binom{n+1}{j} b_j^{(r-1)} - (3r^2 - 1) \sum_{j=0}^n \binom{n}{j} b_j^{(r-1)} + (r^3 - r + 1) \sum_{j=0}^{n-1} \binom{n-1}{j} b_j^{(r-1)} \\ &= (3r - 3r^2 + 1) \sum_{j=0}^{n+1} \binom{n+1}{j} b_j^{(r-1)} + (3r^2 - 1) \sum_{j=0}^{n+1} \binom{n+1}{j} b_j^{(r-1)} \\ &\quad - (3r^2 - 1) \sum_{j=0}^n \binom{n}{j} b_j^{(r-1)} + (r^3 - r + 1) \sum_{j=0}^{n-1} \binom{n-1}{j} b_j^{(r-1)} \\ &= (3r - 3r^2 + 1) \sum_{j=0}^{n+1} \binom{n+1}{j} b_j^{(r-1)} + (3r^2 - 1) \sum_{j=1}^{n+1} \binom{n}{j-1} b_j^{(r-1)} \\ &\quad + (r^3 - r + 1) \sum_{j=0}^{n-1} \binom{n-1}{j} b_j^{(r-1)} \\ &= (3r - 3r^2 + 1) \sum_{j=0}^{n+1} \binom{n+1}{j} b_j^{(r-1)} + (3r - 3r^2 + 1) \sum_{j=1}^{n+1} \binom{n}{j-1} b_j^{(r-1)} \\ &\quad + (6r^2 - 3r - 2) \sum_{j=1}^{n+1} \binom{n}{j-1} b_j^{(r-1)} + (r^3 - r + 1) \sum_{j=0}^{n-1} \binom{n-1}{j} b_j^{(r-1)} \\ &= (3r - 3r^2 + 1) \sum_{j=0}^{n+1} \binom{n+1}{j} b_j^{(r-1)} + (3r - 3r^2 + 1) \sum_{j=0}^{n+1} \binom{n+1}{j} b_{j+1}^{(r-1)} \\ &\quad + (6r^2 - 3r - 2) \sum_{j=1}^{n+1} \binom{n}{j-1} b_j^{(r-1)} + (r^3 - r + 1) \sum_{j=0}^{n-1} \binom{n-1}{j} b_j^{(r-1)} \\ &\quad - (3r - 3r^2 + 1) \sum_{j=1}^{n+1} \binom{n}{j-1} b_{j+1}^{(r-1)}. \end{aligned}$$

From Lemma 2.1, we have

$$\begin{aligned}
RHS &= (3r - 3r^2 + 1) b_{n+2}^{(r)} + (6r^2 - 3r - 2) \sum_{j=0}^n \binom{n}{j} b_{j+1}^{(r-1)} \\
&\quad + (r^3 - r + 1) \sum_{j=0}^{n-1} \binom{n-1}{j} b_j^{(r-1)} + (3r^2 - 3r - 1) \sum_{j=0}^n \binom{n}{j} b_{j+2}^{(r-1)} \\
&= (3r - 3r^2 + 1) b_{n+2}^{(r)} + (r^3 - r + 1) \sum_{j=0}^{n-1} \binom{n-1}{j} b_j^{(r-1)} \\
&\quad + (3r^2 - 3r) \sum_{j=0}^n \binom{n}{j} (b_{j+1}^{(r-1)} + b_{j+2}^{(r-1)}) + (3r^2 - 2) \sum_{j=0}^n \binom{n}{j} b_{j+1}^{(r-1)} \\
&\quad - \sum_{j=0}^n \binom{n}{j} b_{j+2}^{(r-1)} \\
RHS &= b_{n+2}^{(r)} + (3r - 3r^2) b_{n+1}^{(r)} + (r^3 - r + 1) \sum_{j=0}^{n-1} \binom{n-1}{j} b_j^{(r-1)} \\
&\quad + (3r^2 - 2) \sum_{j=1}^n \binom{n-1}{j-1} b_{j+1}^{(r-1)} + (3r^2 - 2) \sum_{j=0}^{n-1} \binom{n-1}{j} b_{j+1}^{(r-1)} \\
&\quad - \sum_{j=0}^n \binom{n}{j} b_{j+2}^{(r-1)} \\
&= b_{n+2}^{(r)} + (3r - 3r^2) b_{n+1}^{(r)} + (3r^2 - 3r) b_n^{(r)} \\
&\quad + (r^3 - 3r^2 + 2r + 1) \sum_{j=0}^{n-1} \binom{n-1}{j} b_j^{(r-1)} + (3r - 2) \sum_{j=0}^{n-1} \binom{n-1}{j} b_{j+1}^{(r-1)} \\
&\quad + (3r^2 - 2) \sum_{j=0}^{n-1} \binom{n-1}{j} b_{j+2}^{(r-1)} - \sum_{j=0}^n \binom{n}{j} b_{j+2}^{(r-1)} \\
&= b_{n+2}^{(r)} + (3r - 3r^2) (b_{n+1}^{(r)} - b_n^{(r)}) \\
&\quad + \sum_{j=0}^{n-1} \binom{n-1}{j} \left[(r^3 - 3r^2 + 2r + 1) b_j^{(r-1)} + (3r - 2) b_{j+1}^{(r-1)} \right. \\
&\quad \quad \left. + (3r^2 - 3) b_{j+2}^{(r-1)} - b_{j+3}^{(r-1)} \right].
\end{aligned}$$

Afterward, by taking account assumption and Lemma 2.1, we deduce

$$\begin{aligned}
RHS &= b_{n+2}^{(r)} + (3r - 3r^2) (b_{n+1}^{(r)} - b_n^{(r)}) \\
&\quad - (3r - 3r^2) \sum_{j=0}^{n-1} \binom{n-1}{j} (b_{j+1}^{(r-1)} + b_{j+2}^{(r-1)}) \\
&= b_{n+2}^{(r)}
\end{aligned}$$

which is completed the proof of this theorem. ■

The characteristic equation of sequences $\{b_n^{(r)}\}$ and $\{c_n^{(r)}\}$ in (2.1) and (2.2) is $\lambda^3 - 3r\lambda^2 + (3r^2 - 1)\lambda - (r^3 - r + 1) = 0$. Let be λ_1 , λ_2 and λ_3 the roots of this equation. Then, binet's formulas of sequences $\{b_n^{(r)}\}$ and $\{c_n^{(r)}\}$ can be expressed as

$$b_n^{(r)} = X_1 \lambda_1^n + Y_1 \lambda_2^n + Z_1 \lambda_3^n, \quad (2.3)$$

and

$$c_n^{(r)} = X_2 \lambda_1^n + Y_2 \lambda_2^n + Z_2 \lambda_3^n, \quad (2.4)$$

where

$$\begin{aligned} X_1 &= \frac{\lambda_1 b_2^{(r)} - (3r\lambda_1 - \lambda_1^2) b_1^{(r)} + (r^3 - r + 1) b_0^{(r)}}{\lambda_1 (\lambda_1 - \lambda_2) (\lambda_1 - \lambda_3)}, \\ Y_1 &= \frac{\lambda_2 b_2^{(r)} - (3r\lambda_2 - \lambda_2^2) b_1^{(r)} + (r^3 - r + 1) b_0^{(r)}}{\lambda_2 (\lambda_2 - \lambda_1) (\lambda_2 - \lambda_3)}, \\ Z_1 &= \frac{\lambda_3 b_2^{(r)} - (3r\lambda_3 - \lambda_3^2) b_1^{(r)} + (r^3 - r + 1) b_0^{(r)}}{\lambda_3 (\lambda_3 - \lambda_1) (\lambda_3 - \lambda_2)}, \end{aligned}$$

and

$$\begin{aligned} X_2 &= \frac{\lambda_1 c_2^{(r)} - (3r\lambda_1 - \lambda_1^2) c_1^{(r)} + (r^3 - r + 1) c_0^{(r)}}{\lambda_1 (\lambda_1 - \lambda_2) (\lambda_1 - \lambda_3)}, \\ Y_2 &= \frac{\lambda_2 c_2^{(r)} - (3r\lambda_2 - \lambda_2^2) c_1^{(r)} + (r^3 - r + 1) c_0^{(r)}}{\lambda_2 (\lambda_2 - \lambda_1) (\lambda_2 - \lambda_3)}, \\ Z_2 &= \frac{\lambda_3 c_2^{(r)} - (3r\lambda_3 - \lambda_3^2) c_1^{(r)} + (r^3 - r + 1) c_0^{(r)}}{\lambda_3 (\lambda_3 - \lambda_1) (\lambda_3 - \lambda_2)}. \end{aligned}$$

Now, we give the sums of iterated binomial transforms for the Padovan and Perrin matrix sequences.

Theorem 2.2 *Sums of sequences $\{b_n^{(r)}\}$ and $\{c_n^{(r)}\}$ are*

$$\begin{aligned} i) \sum_{i=0}^{n-1} b_i^{(r)} &= \frac{b_{n+1}^{(r)} + (1 - 3r) b_n^{(r)} + (r^3 - r + 1) b_{n-1}^{(r)} + (3r - 1) b_0^{(r)} - b_1^{(r)}}{r^3 + 2r} \\ ii) \sum_{i=0}^{n-1} c_i^{(r)} &= \frac{c_{n+1}^{(r)} + (1 - 3r) c_n^{(r)} + (r^3 - r + 1) c_{n-1}^{(r)} + (3r - 1) c_0^{(r)} - c_1^{(r)}}{r^3 + 2r}. \end{aligned}$$

Proof. We omit Padovan case since the proof be quite similar. By considering equation (2.4), we have

$$\sum_{i=0}^{n-1} c_i^{(r)} = \sum_{i=0}^{n-1} (X_2 \lambda_1^n + Y_2 \lambda_2^n + Z_2 \lambda_3^n).$$

Then we obtain

$$\sum_{i=0}^{n-1} c_i^{(r)} = X_2 \left(\frac{\lambda_1^n - 1}{\lambda_1 - 1} \right) + Y_2 \left(\frac{\lambda_2^n - 1}{\lambda_2 - 1} \right) + Z_2 \left(\frac{\lambda_3^n - 1}{\lambda_3 - 1} \right).$$

Afterward, by taking account equations $c_{-1}^{(r)} = 0$, $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = r^3 - r + 1$ and $\lambda_1 + \lambda_2 + \lambda_3 = 3r$, we conclude

$$\sum_{i=0}^{n-1} c_i^{(r)} = \frac{c_{n+1}^{(r)} + (1 - 3r) c_n^{(r)} + (r^3 - r + 1) c_{n-1}^{(r)} + (3r - 1) c_0^{(r)} - c_1^{(r)}}{r^3 + 2r}.$$

■

Theorem 2.3 *The generating functions of the iterated binomial transforms for (\mathcal{P}_n) and (\mathcal{R}_n) are*

$$\begin{aligned} i) \quad \sum_{i=0}^{\infty} b_i^{(r)} x^i &= \frac{b_0^{(r)} + (b_1^{(r)} - 3rb_0^{(r)})x + (b_2^{(r)} - 3rb_1^{(r)} + (3r^2 - 1)b_0^{(r)})x^2}{1 - 3rx + (3r^2 - 1)x^2 - (r^3 - r + 1)x^3}, \\ ii) \quad \sum_{i=0}^{\infty} c_i^{(r)} x^i &= \frac{c_0^{(r)} + (c_1^{(r)} - 3rc_0^{(r)})x + (c_2^{(r)} - 3rc_1^{(r)} + (3r^2 - 1)c_0^{(r)})x^2}{1 - 3rx + (3r^2 - 1)x^2 - (r^3 - r + 1)x^3}. \end{aligned}$$

Proof.

i) Assume that $b(x, r) = \sum_{i=0}^{\infty} b_i^{(r)} x^i$ is the generating function of the iterated binomial transform for (\mathcal{P}_n) . From Theorem 2.1, we obtain

$$\begin{aligned} b(x, r) &= b_0^{(r)} + b_1^{(r)}x + b_2^{(r)}x^2 \\ &\quad + \sum_{i=3}^{\infty} (3rb_{i-1}^{(r)} - (3r^2 - 1)b_{i-2}^{(r)} + (r^3 - r + 1)b_{i-3}^{(r)})x^i \\ &= b_0^{(r)} + b_1^{(r)}x + b_2^{(r)}x^2 + 3rx \sum_{i=3}^{\infty} b_{i-1}^{(r)}x^{i-1} \\ &\quad - (3r^2 - 1)x^2 \sum_{i=3}^{\infty} b_{i-2}^{(r)}x^{i-2} + (r^3 - r + 1)x^3 \sum_{i=3}^{\infty} b_{i-3}^{(r)}x^{i-3} \\ &= b_0^{(r)} + b_1^{(r)}x + b_2^{(r)}x^2 + 3rx \sum_{i=0}^{\infty} b_i^{(r)}x^i - 3rx (b_0^{(r)} + b_1^{(r)}x) \\ &\quad - (3r^2 - 1)x^2 \sum_{i=0}^{\infty} b_i^{(r)}x^i + (3r^2 - 1)x^2 b_0^{(r)} + (r^3 - r + 1)x^3 \sum_{i=0}^{\infty} b_i^{(r)}x^i. \end{aligned}$$

Now rearrangement the equation implies that

$$b(x, r) = \frac{b_0^{(r)} + (b_1^{(r)} - 3rb_0^{(r)})x + (b_2^{(r)} - 3rb_1^{(r)} + (3r^2 - 1)b_0^{(r)})x^2}{1 - 3rx + (3r^2 - 1)x^2 - (r^3 - r + 1)x^3},$$

which equal to the $\sum_{i=0}^{\infty} b_i^{(r)} x^i$ in theorem. Hence the result.

- ii) The proof of generating function of the iterated binomial transform for Perrin matrix sequences can see by taking account proof of i).

■

3 The relationships between new iterated binomial transforms

In this section, we present the relationships between the iterated binomial transform of the Padovan matrix sequence and iterated binomial transform of the Perrin matrix sequence.

Theorem 3.1 *For $n, m \geq 0$, we have*

i) $b_n^{(r)} b_m^{(r)} = b_{n+m}^{(r)}$, where $n \leq m$,

ii) $b_n^{(r)} c_m^{(r)} = c_m^{(r)} b_n^{(r)} = c_{n+m}^{(r)}$,

Proof.

- i) The proof will be done by induction step on r . First of all, for $r = 1$, from the iii) condition of Proposition 1.2, it is true $b_n b_m = b_{n+m}$.

Actually, by assuming the equation in i) holds for all r , that is,

$$\begin{aligned} b_n^{(r)} b_m^{(r)} &= \sum_{i=0}^n \binom{n}{i} b_i^{(r-1)} + \sum_{j=0}^m \binom{m}{j} b_j^{(r-1)} \\ &= \sum_{k=0}^{n+m} \binom{n+m}{k} b_k^{(r-1)} = b_{n+m}^{(r)}. \end{aligned}$$

Then, we need to show that it is true for $r + 1$. That is, From definition of iterated binomial transform, we have

$$\begin{aligned} b_n^{(r+1)} b_m^{(r+1)} &= \left(\sum_{i=0}^n \binom{n}{i} b_i^{(r)} \right) \left(\sum_{j=0}^m \binom{m}{j} b_j^{(r)} \right) \\ &= \left(\sum_{i=0}^n \binom{n}{i} \sum_{k=0}^i \binom{i}{k} b_k^{(r-1)} \right) \left(\sum_{j=0}^m \binom{m}{j} \sum_{l=0}^j \binom{j}{l} b_l^{(r-1)} \right). \end{aligned}$$

And, by considering assumption, we obtain

$$\begin{aligned}
b_n^{(r+1)}b_m^{(r+1)} &= \sum_{i=0}^n \sum_{j=0}^m \binom{n}{i} \binom{m}{j} b_{i+j}^{(r)} \\
&= \binom{n}{0} \binom{m}{0} b_0^{(r)} + \binom{n}{0} \binom{m}{1} b_1^{(r)} + \cdots + \binom{n}{0} \binom{m}{m} b_m^{(r)} \\
&\quad + \binom{n}{1} \binom{m}{0} b_1^{(r)} + \binom{n}{1} \binom{m}{1} b_2^{(r)} + \cdots + \binom{n}{1} \binom{m}{m} b_{m+1}^{(r)} + \\
&\quad \vdots \\
&\quad + \binom{n}{n} \binom{m}{0} b_n^{(r)} + \binom{n}{n} \binom{m}{1} b_{n+1}^{(r)} + \cdots + \binom{n}{n} \binom{m}{m} b_{n+m}^{(r)} \\
&= \binom{n}{0} \binom{m}{0} b_0^{(r)} + \left[\binom{n}{0} \binom{m}{1} + \binom{n}{1} \binom{m}{0} \right] b_1^{(r)} \\
&\quad + \left[\binom{n}{0} \binom{m}{2} + \binom{n}{1} \binom{m}{1} + \binom{n}{2} \binom{m}{0} \right] b_2^{(r)} + \cdots \\
&\quad + \left[\binom{n}{0} \binom{m}{k} + \binom{n}{1} \binom{m}{k-1} + \cdots + \binom{n}{k} \binom{m}{0} \right] b_k^{(r)} + \cdots \\
&\quad + \binom{n}{n} \binom{m}{m} b_{n+m}^{(r)}.
\end{aligned}$$

By taking account Vandermonde identity $\sum_{j=0}^k \binom{x}{j} \binom{y}{k-j} = \binom{x+y}{k}$, we get

$$\begin{aligned}
b_n^{(r+1)}b_m^{(r+1)} &= \binom{n+m}{0} b_0^{(r)} + \binom{n+m}{1} b_1^{(r)} + \binom{n+m}{2} b_2^{(r)} + \cdots \\
&\quad + \binom{n+m}{k} b_k^{(r)} + \cdots + \binom{n+m}{n+m} b_{n+m}^{(r)} \\
&= \sum_{i=0}^{n+m} \binom{n+m}{i} b_i^{(r)} \\
&= b_{n+m}^{(r+1)}.
\end{aligned}$$

ii) The proof is similar proof of i).

■

Theorem 3.2 *The properties of the transforms $\{b_n^{(r)}\}$ and $\{c_n^{(r)}\}$ would be illustrated by following way:*

i) $b_{n+1}^{(r)} - b_n^{(r)} = b_1^{(r-1)} b_n^{(r)},$

ii) $c_{n+1}^{(r)} - c_n^{(r)} = b_1^{(r-1)} c_n^{(r)},$

$$iii) \quad c_{n+1}^{(r)} - c_n^{(r)} = c_1^{(r-1)} b_n^{(r)}.$$

Proof. We will omit the proof of *ii*) and *iii*), since it is quite similar with *i*). Therefore, by considering definition of iterated binomial transform and Lemma 2.1-*i*), we have

$$b_{n+1}^{(r)} - b_n^{(r)} = \sum_{i=0}^n \binom{n}{i} b_{i+1}^{(r-1)}.$$

From Theorem 3.1-*i*), we get

$$b_{n+1}^{(r)} - b_n^{(r)} = \sum_{i=0}^n \binom{n}{i} b_i^{(r-1)} b_1^{(r-1)} = b_1^{(r-1)} b_n^{(r)}.$$

■

Theorem 3.3 For $n, m \geq 0$, the relation between the transforms $\{b_n^{(r)}\}$ and $\{c_n^{(r)}\}$ is

$$c_m^{(r-1)} b_n^{(r)} = b_m^{(r-1)} c_n^{(r)}.$$

Proof. By considering definition of iterated binomial transform, we have

$$\begin{aligned} c_m^{(r-1)} b_n^{(r)} &= c_m^{(r-1)} \sum_{i=0}^n \binom{n}{i} b_i^{(r-1)} \\ &= \sum_{i=0}^n \binom{n}{i} c_m^{(r-1)} b_i^{(r-1)}. \end{aligned}$$

From Theorem 3.1-*ii*), we get

$$\begin{aligned} c_m^{(r-1)} b_n^{(r)} &= \sum_{i=0}^n \binom{n}{i} c_{m+i}^{(r-1)} \\ &= \sum_{i=0}^n \binom{n}{i} b_m^{(r-1)} c_i^{(r-1)} \\ &= b_m^{(r-1)} c_n^{(r)}. \end{aligned}$$

■

By choosing $m = 0$ in Theorem 3.3 and using the initial conditions of equations (2.1) and (2.2), we obtain the following corollary.

Corollary 3.1 The following equalities are hold:

$$i) \quad c_n^{(r)} = \mathcal{R}_0 b_n^{(r)},$$

$$ii) \quad b_n^{(r)} = \mathcal{R}_0^{-1} c_n^{(r)}.$$

Corollary 3.2 *We should note that choosing $r = 1$ in the all results of Section 2 and 3, it is actually obtained some properties of the iterated binomial transforms for Padovan and Perrin matrix sequences such that the recurrence relations, Binet formulas, summations, generating functions and relationships of between binomial transforms for Padovan and Perrin matrix sequences.*

Conclusion 3.1 *In this paper, we define the iterated binomial transforms for Padovan and Perrin matrix sequences and present some properties of these transforms. By the results in Sections 2 and 3 of this paper, we have a great opportunity to compare and obtain some new properties over these transforms. This is the main aim of this paper. Thus, we extend some recent result in the literature.*

In the future studies on the iterated binomial transform for number sequences, we expect that the following topics will bring a new insight.

- (1) *It would be interesting to study the iterated binomial transform for Fibonacci and Lucas matrix sequences,*
- (2) *Also, it would be interesting to study the iterated binomial transform for Pell and Pell-Lucas matrix sequences.*

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